Problem set 02

Nonlinear Optics for Quantum Technologies

February 27, 2025

1 Pulse propagation

Definitions: For this exercise we will define the fourier transform as:

$$f(\omega) = \int f(t)e^{i\omega t}dt$$

$$f(t) = \int f(\omega)e^{-i\omega t}\,\frac{d\omega}{2\pi}$$

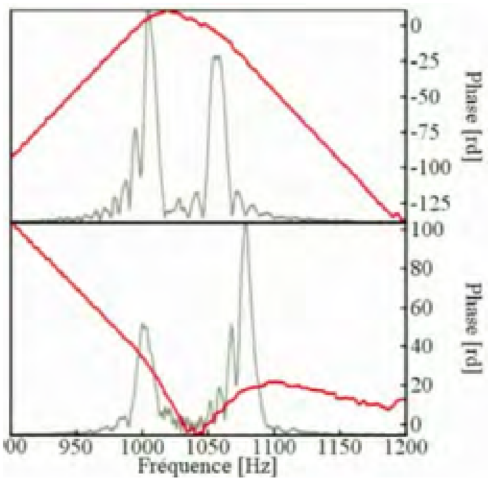
We further recall the **Parseval Identity** for two functions f and g:

$$\int_{-\infty}^{\infty} f^*(t)g(t)dt = \int_{-\infty}^{\infty} f^*(\omega)g(\omega)\,\frac{d\omega}{2\pi}$$

Let us now consider a pulse with normalised envelope $f(t)$ such that:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(\omega)|^2\,\frac{d\omega}{2\pi} = 1$$

1. Consider the spectral representation of the pulse $f(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$. What is the effect of a translation of time $t \to t + \tau$ on $f(\omega)$?
2. We write $f(\omega) = |f(\omega)|e^{i\phi(\omega)}$ where $\phi(\omega)$ is the spectral phase. The two graphs below show the Fourier transform of an acoustic signal consisting of two notes (B,C). In which order where they played?



3. Using the decomposition $f(\omega) = |f(\omega)|e^{i\phi(\omega)}$ write down the derivative $f'(\omega) = \frac{df}{d\omega}$
4. We are now interested in the center of the pulse in the time domain, which is described by the mean value (barycenter)

$$\langle t \rangle = \int_{-\infty}^{\infty} t|f(t)|^2 dt$$

Explain why the dimension of the right-hand term is indeed time t and not t^2. Then show that

$$\langle t \rangle = \left\langle \frac{d\phi}{d\omega} \right\rangle$$

From this result we will call the quantity

$$\tau_g(\omega) = \frac{d\phi}{d\omega}$$

the **group delay**. The barycenter of a pulse $f(t)$ is therefore $\langle t \rangle = \langle \tau_g(\omega) \rangle$ where the averages are weighted by $|f(t)|^2$ and $|f(\omega)|^2$, respectively.

5. Now, consider a light pulse propagating along z, whose complex expression (positive part of the Fourier transform) is $E(z, \omega)$. At $z = 0$ we have $E(0, \omega) = \mathrm{E}_0 f(0, \omega)$, where $f(z, \omega)$ is normalized. Write down $E(z, \omega)$ in a medium with dispersion $k(\omega)$.

6. How is the spectral phase $\phi(z, \omega)$ evolving along z with respect to $\phi(0, \omega)$?

7. Using the result from 4), show that $\langle t \rangle_z = \langle t \rangle_0 + \frac{1}{v_g} z$ and give the expression of $\frac{1}{v_g}$.

8. What should be the form of $n(\omega)$ (index refraction) so that v_g is constant over some frequency range? What happens to a short light pulse propagating in such a medium?

9. We now assume that to a first approximation $n(\omega)$ increases linearly with ω (locally, close to ω_0).

   - Discuss qualitatively what would happen to a short pulse centered at ω_0 propagating in such medium?
   - In a Taylor expansion to lowest order of $k(\omega)$ around ω_0, what terms govern the speed of the pulse center and its duration?
   - Explain why a Fourier transform limited pulse going through a dispersive medium is being both stretched (in time) and "chirped"? (Chirp refers to the 'instantaneous' frequency changing over time)

2 First order degree of coherence

Notations:
We define the "analytical signals" $E^+(\vec{r},t)$, $E^-(\vec{r},t)$ as

$$E(r,t) = E^+(\vec{r},t) + E^-(\vec{r},t)$$

with

$$E^+(\vec{r},t) = \int_0^\infty E(\vec{r},\omega)e^{-i\omega t}\frac{d\omega}{2\pi}$$

and

$$E^-(\vec{r},t) = \int_0^\infty E(\vec{r},-\omega)e^{+i\omega t}\frac{d\omega}{2\pi} = E^{+*}(\vec{r},t)$$

For a stationary but stochastic (i.e. fluctuating signal) we define the unnormalized autocorrelation function at a particular point in space

$$G^{(1)}(\tau) =< E^-(0)E^+(\tau) >= \int_{-\infty}^{\infty} E^-(t)E^+(t+\tau)dt$$

1. Using the Cauchy-Schwarz inequality

$$|\langle f(x)g(y)\rangle|^2 \leq \ \langle |f(x)|^2\rangle\langle |g(y)|^2\rangle$$

give an upper bound to the value of the **first order degree of coherence** defined as:

$$g^{(1)}(\tau) = \frac{G^{(1)}(\tau)}{G^{(1)}(0)}$$

2. Express the intensity at the output of a Mach Zehnder interferometer $\langle I(\tau)\rangle$ as a function of $g^{(1)}(\tau)$ where $\tau = \frac{\delta z}{c}$ is the time delay due to the path difference δz between the two arms of the interferometer. Assume a 50/50 beamsplitter.

3. The (first order) coherence time is defined as:

$$T_{coh} = \int_{-\infty}^{\infty} |g^{(1)}(\tau)|^2 d\tau$$

What is the value of T_{coh} in the following case

- Homogenous broadening $g^{(1)}(\tau) = e^{-i\omega_0\tau-\gamma|\tau|}$
- Inhomogenous broadening $g^{(1)}(\tau) = e^{-i\omega_0\tau-\frac{1}{2}\delta^2\tau^2}$

4. The power spectral density (PSD) associated with the signal $E(t)$ is

$$PSD(\omega) = |E(\omega)|^2$$

where $E(\omega)$ is the Fourier transform (as defined in the previous exercise) of $E(t)$

- Show that $PSD(\omega) = \int_{-\infty}^{\infty} G^{(1)}(\tau)e^{i\omega\tau}d\tau$, i.e. the PSD is the Fourier transform of the signal's first order coherence.
- What is the lineshape of the normalized PSD for the two types of broadening considered in 4?
- Explain how you can measure the PSD (or spectrum) of a light source using a Mach-Zehnder interferometer as in question 2.
- How can you increase the coherence time of a light source?

Further discussion points for the interested students, (will not be further treated in the exercise class):

- Consider a Fourier transform limited pulse as in question 1.9, with pulse duration t_p. Calculate its first order coherence time. Does its coherence time increase over time due to propagation in a medium with dispersion? Do you think a linear interferometer is an adequate instrument to measure a pulse duration?
- In exercise 2 we did not consider the spatial component of coherence. Similar to the coherence time of 2.3 it is possible to define a coherence length for a light source. How can you measure the spatial coherence of a light beam?
- Do you think fermions such as electrons also display first order coherence?